

Quantum-mechanical description of Lense-Thirring effect for relativistic scalar particles

A.J. Silenko*

*Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna, Russia
Research Institute for Nuclear Problems,
Belarusian State University, Minsk, Belarus*

Abstract

Exact expression for the Foldy-Wouthuysen Hamiltonian of scalar particles is used for a quantum-mechanical description of the relativistic Lense-Thirring effect. The exact evolution of the angular momentum operator in the Kerr field approximated by a spatially isotropic metric is found. The quantum-mechanical description of the full Lense-Thirring effect based on the Laplace-Runge-Lenz vector is given in the nonrelativistic and weak-field approximation. Relativistic quantum-mechanical equations for the velocity and acceleration operators are obtained. The equation for the acceleration defines the Coriolis-like and centrifugal-like accelerations and presents the quantum-mechanical description of the frame-dragging effect.

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*Electronic address: alsilenko@mail.ru

I. INTRODUCTION

The well-known Lense-Thirring (LT) effect [1] is a gravitomagnetic effect of frame-dragging predicted by general relativity. It consists in secular precessions of the longitude of the ascending node and the argument of pericenter of a test particle freely orbiting a central spinning mass endowed with angular momentum. This effect also manifests in a precession of the orbit and in a Coriolis-like force acting on the moving particle.

The description of a spinless particle in a Riemannian spacetime of general relativity is based on the covariant Klein-Gordon-Fock equation [2] added by an appropriate term describing a nonminimal coupling to the scalar curvature and conserving the conformal invariance of the equation for a massless scalar particle [3, 4]. The inclusion of the Penrose-Chernikov-Tagirov term has been argued for both massive and massless particles [4].

Accioly and Blas [5] have brought the initial equation to the Hamiltonian form and have performed the exact Foldy-Wouthuysen (FW) transformation of the Hamiltonian obtained. They have considered a massive particle in a static isotropic metric. The transformation method used in Ref. [5] is inapplicable to massless particles and does not cover nonstatic spacetimes. As a result, an information about a specific manifestation of the conformal invariance in the FW representation has not been obtained.

The generalized method of transformation of the Klein-Gordon-Fock equation to the Hamiltonian form useful for both massive and massless particles has been developed in Ref. [6]. Its application in Ref. [7] has allowed to fulfill the FW transformation and to prove the conformal invariance of the relativistic FW Hamiltonian for a wide class of inertial and gravitational fields. General quantum-mechanical equations of motion have been derived and their classical limit have been obtained.

In the present work, the exact FW Hamiltonian for a scalar particle in the Kerr field approximated by a spatially isotropic metric [7] is used for a quantum-mechanical description of the relativistic LT effect. We obtain the relativistic equation of motion for the angular momentum operator, perform the quantum-mechanical description of the LT effect in the nonrelativistic and weak-field approximation, and derive quantum-mechanical equations for the velocity and acceleration operators. The results obtained are compared with the classical description.

II. FOLDY-WOUTHUYSEN HAMILTONIAN AND EQUATIONS OF MOTION

The initial covariant Klein-Gordon-Fock equation with the additional term [3, 4] describes a scalar particle in a Riemannian spacetime and is given by

$$(\square + m^2 - \lambda R)\psi = 0, \quad \square \equiv \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu. \quad (1)$$

The Penrose-Chernikov-Tagirov coupling is defined by $\lambda = 1/6$. This choice of λ has been unambiguously confirmed in Refs. [5, 7]. Sign of λ depends on the definition of R . In the present work, the signature is $(+ - - -)$ and the Ricci scalar curvature is defined by $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\alpha_{\mu\alpha\nu}$, where $R^\alpha_{\mu\beta\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \dots$ is the Riemann curvature tensor.

The generalized Feshbach-Villars transformation [6] and the subsequent nonunitary one allow to represent Eq. (1) in the Hamiltonian form describing both massive and massless particles [7]:

$$\begin{aligned} \mathcal{H}' &= \rho_3 \frac{N^2 + T'}{2N} + i\rho_2 \frac{-N^2 + T'}{2N} - i\Upsilon', \\ T' &= \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \frac{m^2 - \lambda R}{g^{00}} + \frac{1}{f} \nabla_i (\sqrt{-g} G^{ij}) \nabla_j \left(\frac{1}{f} \right) \\ &\quad + \sqrt{\frac{\sqrt{-g}}{g^{00}}} G^{ij} \nabla_i \nabla_j \left(\frac{1}{f} \right) + \frac{1}{4f^4} [\nabla_i (\Gamma^i)]^2 \\ &\quad - \frac{1}{2f^2} \nabla_i \left(\frac{g^{0i}}{g^{00}} \right) \nabla_j (\Gamma^j) - \frac{g^{0i}}{2g^{00}f^2} \nabla_i \nabla_j (\Gamma^j), \\ \Upsilon' &= \frac{1}{2} \left\{ \partial_i, \frac{g^{0i}}{g^{00}} \right\}, \quad G^{ij} = g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}, \quad \Gamma^i = \sqrt{-g}g^{0i}, \quad f = \sqrt{g^{00}\sqrt{-g}}, \end{aligned} \quad (2)$$

where the nabla operators act only on the operators in brackets and the primes denote nonunitary transformed operators. Equation (2) is exact and covers any inertial and gravitational fields.

The sufficient condition of the exact FW transformation [6, 8, 9] applied to scalar particles is given by $\partial_0 T' - [T', \Upsilon'] = 0$. When this condition is satisfied, the exact FW Hamiltonian reads [7]

$$\mathcal{H}_{FW} = \rho_3 \sqrt{T'} - i\Upsilon'. \quad (3)$$

This equation covers *all static spacetimes* ($\Upsilon' = 0$) and some important cases of stationary ones.

The metric of the rotating Kerr source has been reduced to the Arnowitt-Deser-Misner form [10] by Hergt and Schäfer [11]. This form reproduces the Kerr solution only approximately. The form of the metric can be additionally simplified due to an introduction of

spatially isotropic coordinates and dropping terms violating the isotropy [12]:

$$ds^2 = V^2(dx^0)^2 - W^2\delta_{ij}(dx^i - K^i dx^0)(dx^j - K^j dx^0), \quad \mathbf{K} = \boldsymbol{\omega} \times \mathbf{r}. \quad (4)$$

The use of the approximate Kerr metric allows to fulfill the *exact* FW transformation when V, W , and $\boldsymbol{\omega}$ depend only on the isotropic radial coordinate r . In this approximation, the metric is defined by

$$V(r) = \frac{1 - \mu/(2r)}{1 + \mu/(2r)} + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right), \quad W(r) = \left(1 + \frac{\mu}{2r}\right)^2 + \mathcal{O}\left(\frac{\mu a^2}{r^3}\right), \quad (5)$$

$$\boldsymbol{\omega}(r) = \frac{2\mu c}{r^3} \mathbf{a} \left[1 - \frac{3\mu}{r} + \frac{21\mu^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right)\right].$$

Here $\mathbf{a} = \mathbf{J}/(Mc)$, $\mu = GM/c^2$; the total mass M and the total angular momentum \mathbf{J} (directed along the z axis) define the Kerr source uniquely. The leading term in the expression for $\boldsymbol{\omega}(r) = \omega(r)\mathbf{e}_z$ corresponds to the LT approximation.

We can pass on from the Kerr field approximated by Eqs. (4),(5) to a frame rotating in this field with the angular velocity \mathbf{o} after the transformation $dx^i \rightarrow dX^i = dx^i + (\mathbf{o} \times \mathbf{r})dx^0$. The stationary metric of this frame can be obtained from Eqs. (4),(5) with the replacement $\boldsymbol{\omega} \rightarrow \boldsymbol{\Omega} = \boldsymbol{\omega} - \mathbf{o}$. In particular, it covers an observer on the ground of a rotating source like the Earth or on a satellite. In this case, $\mathbf{o} = \mathbf{J}/I$, where I is the moment of inertia. It should be taken into account that frames rotating in the isotropic and Cartesian coordinates are not equivalent. The exact FW Hamiltonian is given by Eq. (3). When $\lambda = 1/6$, the operators T' and Υ' are defined by [7]

$$T' = m^2 V^2 + \mathcal{F} \mathbf{p}^2 \mathcal{F} - \frac{1}{4} \nabla \mathcal{F} \cdot \nabla \mathcal{F} + \frac{1}{6} \mathcal{F} \Delta \mathcal{F} + \frac{1}{12} (x^2 + y^2) (\Omega'_r)^2, \quad (6)$$

$$-i\Upsilon' = \boldsymbol{\Omega} \cdot (\mathbf{r} \times \mathbf{p}), \quad \mathcal{F} = \frac{V}{W},$$

and derivatives with respect to r are denoted by indexes. \rightarrow In particular, for the LT metric

$$\boldsymbol{\Omega}(r) = \frac{2G\mathbf{J}}{c^2 r^3}, \quad V(r) = 1 - \frac{GM}{c^2 r}, \quad W(r) = 1 + \frac{GM}{c^2 r}. \quad (7)$$

The quantum-mechanical equations of motion in the FW representation defining the force, velocity, and acceleration read ($p_0 \equiv \mathcal{H}_{FW}$)

$$F^i \equiv \frac{dp^i}{dt} = \frac{\partial p^i}{\partial t} + \frac{i}{\hbar} [\mathcal{H}_{FW}, p^i] = \frac{1}{2} \frac{\partial}{\partial t} \{g^{i\mu}, p_\mu\} + \frac{i}{2\hbar} [\mathcal{H}_{FW}, \{g^{i\mu}, p_\mu\}], \quad (8)$$

$$\mathcal{V}^i \equiv \frac{dx^i}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, x^i], \quad \mathcal{W}^i = \frac{\partial \mathcal{V}^i}{\partial t} + \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathcal{V}^i].$$

Any commutation adds the factor \hbar as compared with the product of operators.

It has been proved in Ref. [13] that satisfying the condition of the Wentzel-Kramers-Brillouin approximation allows to use this approximation in the relativistic case and to obtain a classical limit of the relativistic quantum mechanics. Determination of the classical limit reduces to the replacement of operators in the FW Hamiltonian and quantum-mechanical equations of motion in the FW representation by respective classical quantities. The classical limit of the general FW Hamiltonian is given by [7]

$$H = \left(\frac{m^2 - G^{ij} p_i p_j}{g^{00}} \right)^{1/2} - \frac{g^{0i} p_i}{g^{00}}. \quad (9)$$

It coincides with the classical Hamiltonian derived in Ref. [14].

The classical limit of Eq. (8) reads

$$\begin{aligned} \mathcal{V}^i &= \frac{G^{ij} p_j}{\sqrt{g^{00}(m^2 - G^{ij} p_i p_j)}} + \frac{g^{0i}}{g^{00}}, \\ F^i &= p_\mu \frac{\partial g^{i\mu}}{\partial t} + g^{0i} \frac{\partial H}{\partial t} + g^{ij} \partial_j H + p_\mu \mathcal{V}^j \partial_j g^{i\mu}. \end{aligned} \quad (10)$$

It coincides with the corresponding classical equations which follow from Hamiltonian (9) and the Hamilton equations. Thus, the quantum-mechanical and classical equations are in the best compliance.

For example, the exact metric of a general noninertial frame characterized by the acceleration \mathbf{a} and the rotation \mathbf{o} of an observer is defined by $V = 1 + \mathbf{a} \cdot \mathbf{r}$, $W = 1$, $\mathbf{\Omega} = -\mathbf{o}$ [15]. In this case, the classical limit of the Hamiltonian and equations of motion is given by [7]

$$\begin{aligned} H &= (1 + \mathbf{a} \cdot \mathbf{r}) \sqrt{m^2 + \mathbf{p}^2} - \mathbf{o} \cdot (\mathbf{r} \times \mathbf{p}), \quad \mathcal{V} = (1 + \mathbf{a} \cdot \mathbf{r}) \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} - \mathbf{o} \times \mathbf{r}, \\ \mathcal{W} &= -\mathbf{a}(1 + \mathbf{a} \cdot \mathbf{r}) - 2\mathbf{o} \times \mathcal{V} - \mathbf{o} \times (\mathbf{o} \times \mathbf{r}) + \frac{2\mathbf{a} \cdot \mathcal{V} + \mathbf{a} \cdot (\mathbf{o} \times \mathbf{r})}{1 + \mathbf{a} \cdot \mathbf{r}} (\mathcal{V} + \mathbf{o} \times \mathbf{r}), \end{aligned} \quad (11)$$

where $\mathbf{p} \equiv (-p_1, -p_2, -p_3)$. Leading terms in Eq. (11) reproduce well-known classical results [16].

III. QUANTUM-MECHANICAL DESCRIPTION OF THE LENSE-THIRING EFFECT

The results obtained allow to derive quantum-mechanical equations describing the LT effect. When a metric depends only on r , it is convenient to consider the evolution of the

angular momentum operator $\mathbf{l} = \mathbf{r} \times \mathbf{p}$. Dynamics of this operator in a frame rotating in the Kerr field approximated by a spatially isotropic metric is defined by

$$\frac{d\mathbf{l}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{l}] = \mathbf{\Omega} \times \mathbf{l}, \quad \mathbf{\Omega} = \boldsymbol{\omega} - \boldsymbol{o}. \quad (12)$$

Since the operators $\mathbf{\Omega}$ and \mathbf{l} commute, this equation is exact *for the chosen metric*.

The quantity $\boldsymbol{\omega}$ characterizes an evolution of the longitude of the ascending node, Υ : $\Upsilon = \Upsilon_0 + \omega t$. Equations (5),(12) provide for a relativistic post-Newtonian description of this evolution:

$$\omega = \frac{2GJ}{c^2 r^3} \left[1 - \frac{3GM}{c^2 r} + \frac{21G^2 M^2}{4c^4 r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right) \right]. \quad (13)$$

This is a part of the LT effect. The longitude of the ascending node can be measured and its measurement is important for astrophysics.

A transition to the classical limit [13] and a calculation of the period average in the nonrelativistic and weak-field approximation results in

$$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{b^3(1-e^2)^{3/2}}, \quad \omega = \frac{2GJ}{c^2 b^3(1-e^2)^{3/2}}, \quad (14)$$

where b is the semimajor axis and e is the eccentricity.

The quantum-mechanical description of the full LT effect is based on the Laplace-Runge-Lenz vector. In this case, we confine ourselves by the nonrelativistic and weak-field approximation. The operator form of the Laplace-Runge-Lenz vector is given by

$$\mathbf{A} = \frac{1}{2} (\mathbf{p} \times \mathbf{l} - \mathbf{l} \times \mathbf{p}) - mk\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}, \quad k = GMm. \quad (15)$$

The nonrelativistic FW Hamiltonian for the Kerr field in the LT approximation reads

$$\mathcal{H}_{FW} = \rho_3 \left(mc^2 - \frac{k}{r} + \frac{\mathbf{p}^2}{2m} \right) + \mathbf{\Omega} \cdot \mathbf{l}, \quad (16)$$

where $\mathbf{\Omega}$ is defined by Eq. (7). The precession of pericenter of the orbit is defined by the commutator of the operators \mathcal{H}_{FW} and \mathbf{A} :

$$\frac{d\mathbf{A}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \mathbf{A}] = \frac{1}{2} (\mathbf{\Omega} \times \mathbf{A} - \mathbf{A} \times \mathbf{\Omega}) + \frac{3G}{2c^2} \left\{ \frac{\mathbf{J} \cdot \mathbf{l}}{r^5}, (\mathbf{r} \times \mathbf{l} - \mathbf{l} \times \mathbf{r}) \right\}. \quad (17)$$

The transition to the classical limit and the calculation of the period average leads to the LT equation:

$$\frac{d\mathbf{A}}{dt} = \mathbf{\Omega}_{LT} \times \mathbf{A}, \quad \mathbf{\Omega}_{LT} = \frac{2G}{c^2 r^3} \left[\mathbf{J} - 3(\mathbf{J} \cdot \hat{\mathbf{l}})\hat{\mathbf{l}} \right], \quad (18)$$

where $\hat{\mathbf{l}} = \mathbf{l}/l$.

The existence of the frame dragging can also be shown. If we hold only main terms in the *relativistic* FW Hamiltonian presented by Eqs. (3),(6), the velocity operator in the field defined by the LT metric (7) is given by

$$\boldsymbol{\mathcal{V}} = \frac{\rho_3}{2} \left\{ \frac{c}{\sqrt{m^2 c^2 V^2 + \mathcal{F} \mathbf{p}^2 \mathcal{F}}}, \mathcal{F} \mathbf{p} \mathcal{F} \right\} + \boldsymbol{\Omega} \times \mathbf{r}. \quad (19)$$

In the weak-field approximation, the part of the acceleration operator defined only by the rotation of the source is equal to

$$\boldsymbol{\mathcal{W}} = \boldsymbol{\Omega} \times \boldsymbol{\mathcal{V}} - \boldsymbol{\mathcal{V}} \times \boldsymbol{\Omega} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \quad (20)$$

This equation defines the Coriolis-like and centrifugal-like accelerations and therefore describes the quantum-mechanical frame-dragging effect.

It is important that the classical limit of all obtained quantum-mechanical equations coincides with the corresponding classical equations.

IV. CONCLUSIONS

The use of the exact FW Hamiltonian for scalar particles in the frame rotating in the Kerr field approximated by a spatially isotropic metric [7] has allowed us to fulfill the detailed quantum-mechanical description of the relativistic LT effect. The exact evolution of the angular momentum operator in the Kerr field approximated by a spatially isotropic metric is found. The quantum-mechanical equation defining the precession of pericenter of the orbit (full LT effect) is based on the Laplace-Runge-Lenz vector and derived in the nonrelativistic and weak-field approximation. Relativistic quantum-mechanical equations for the velocity and acceleration operators are obtained. The equation for the acceleration defines the Coriolis-like and centrifugal-like accelerations and presents the quantum-mechanical description of the frame-dragging effect.

The classical limit of the derived quantum-mechanical equations coincides with corresponding classical ones. This important conclusion confirms the general statement made in Refs. [4, 7, 17] and unambiguously shows a deep connection between the relativistic quantum mechanics of scalar particles in Riemannian spacetimes and the classical general relativity.

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